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Energy decay rate for a quasi-linear wave equation with localized strong dissipation

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ABSTRACT

The purpose of this paper is to derive a sharp energy decay estimate for a quasi-linear wave equation with localized strong dissipation of the type $-\nabla\cdot(a(x)\nabla u_t)$ in a domain Ω of $\mathbb{R}^N$, where $a(x)$ is a nonnegative function supported only on a part of the boundary $\partial\Omega$. We note that the index of algebraic decay depends on dimension N and no geometrical condition is imposed on the boundary $\partial\Omega$.

1. Introduction

Many authors have studied the global existence of solutions and the energy decay estimate of quasi-linear wave equations with some dissipative term [1–3]. Ebihara [4] discussed the global existence and the decay for nonlinear evolution equations with strong dissipation under small initial data. Nakao [5] studied the energy decay for quasi-linear wave equations with strong viscosity. Ono [6] has proved the global existence, decay estimates and blow up results for the degenerate nonlinear wave equation of Kirchhoff type with strong dissipation. Also, many authors dealt with the global existence of solutions to some wave equations with localized dissipation [7–10]. Motivated by these papers, we will consider the initial boundary value problem related to a quasi-linear wave equation with a special dissipative term, which satisfies a localized dissipative property and is weaker than the fully strong dissipative property:

$$u_{tt}-\nabla\cdot\{\sigma(|\nabla u|^2)\nabla u\}-\nabla\cdot(a(x)\nabla u_t)=0\quad \text{in } \Omega\times[0,\infty), \tag{1.1}$$

$$u=0\quad \text{on } \partial\Omega\times[0,\infty), \tag{1.2}$$

$$u(x,0)=u_0(x),\qquad u_t(x,0)=u_1(x)\quad \text{in } \Omega, \tag{1.3}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$, $\sigma(v)$ is a function like $\sigma(v)=1/\sqrt{1+v}$ and $a(x)$ is a nonnegative smooth function on $\overline{\Omega}$. In particular, we concentrate on localized degenerate dissipation that is active in only a localized area on the boundary.

The main goal of this paper is to show that the energy for problem (1.1)–(1.3) decay to zero algebraically when $t\to\infty$ and derive precise energy decay estimates for problem (1.1)–(1.3). We note that the energy decay rate depends on the degeneracy of $a(x)$ as well as the regularity of the solution itself. Furthermore, we would like to emphasize that our results and technical methods can also be applied to another type (for example, a Kirchhoff type equation) of quasi-linear equations as well as some practical models, for example, the acoustic wave model and the minimal surface model (see [11,12]). For some related works involving the energy decay for the wave equation with localized dissipation, see [13–17,9].

The paper is organized as follows. In Section 2, we state our main results. Section 3 is devoted to the proof of our main result.

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2. Preliminaries and the main result

Throughout this paper, we shall denote Sobolev spaces $H^m(\Omega)$ with respect to the norm

$$\|f\|_{H^m(\Omega)} = \sum_{|\alpha|\le m} \|D^\alpha f\|_{L^2(\Omega)},$$

where $D^\alpha = \partial^{|\alpha|}/(\partial x_1^{\alpha_1}, \dots, \partial x_N^{\alpha_N})$, $\alpha = (\alpha_1, \dots, \alpha_N)$, $\alpha_i \ge 0$, $i = 1, \dots, N$, $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_N$, $\|\cdot\|_{L^p(\Omega)}$ denotes L^p norm on an open set $\Omega \in \mathbb{R}^N$ and $\|\cdot\| = \|\cdot\|_{L^2(\Omega)}$. The space $H_0^m(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $H^m(\Omega)$. Now, let us give precise estimates of $a(x)$ and $\sigma(v)$.

Hypothesis I. $a(x) \ge 0$ on Ω, $a(\cdot)$ belong to $C^{m-1}(\bar{\Omega})$ for some integer $m > 0$ and

$$a(x) > 0 \quad \text{a.e.,} \;\; x \in \omega \text{ and } \int_\omega a(x)^{-p} \mathrm{d}x < \infty \tag{2.1}$$

for some $p \in (0, \infty)$ and an open set $\omega \subset \bar{\Omega}$ including the part $\Gamma(x_0)$ of $\partial\Omega$. Here $\Gamma(x_0)$ is the part of the boundary $\partial\Omega$ given by $\Gamma(x_0) = \{x \in \partial\Omega \,|\, (x - x_0)\cdot \nu(x) > 0\}$, where $x_0 \in \mathbb{R}^N$ and $\nu(x)$ denotes the outward unit normal vector at $x \in \partial\Omega$.

Hypothesis II. $\sigma(v)$ belongs to $C^{m+1}(\mathbb{R}^+ = [0, \infty))$ and satisfies the condition that there exists $L > 0$ and $l_0 \equiv l_0(L) > 0$ such that $\sigma(v) \ge l_0 > 0$ and $\sigma(v) + 2\sigma'(v)v \ge l_0 > 0$ for v, $0 \le v \le L$.

We introduce the following well-known lemmas, which will be used in the proof of the main result. We need the following lemma of Nakao [5].

Lemma 2.1. *Let $\phi(t)$ be a nonnegative function on $\mathbb{R}^+$ satisfying*

$$\sup_{t\le s\le t+T} \phi(s)^{1+\gamma} \le C\{\phi(t) - \phi(t+T)\}$$

with $T > 0$, $\gamma > 0$ and C is some positive constant. Then $\phi(t)$ has the decay property

$$\phi(t) \le \left\{\phi(0)^{-\gamma} + \frac{\gamma}{C}(t - T)\right\}^{-\frac{1}{\gamma}}$$

for $t \ge T$.

The existence and regularity of solutions u to problem (1.1)–(1.3) are given by the following result (see [8]):

Theorem 2.1. *Let m be a positive integer and (u_0, u_1) satisfy the compatibility condition of the mth order. Suppose that Hypotheses I and II are satisfied with m and $p > 0$ such that*

$$m \ge \frac{N}{2} + 2 \quad \text{and} \quad (2m - N - 1)\,p > N.$$

Then there exists an open unbounded set ζ including $(0, 0)$ in D_m^0 such that if $(u_0, u_1) \in \zeta$, problem (1.1)–(1.3) admits a unique solution $u(t)$ in the class

$$X_m = \cap_{k=0}^m C^k([0, \infty); H_{m+1-k}(\Omega) \cap H_0^1(\Omega)) \cap C^{m+1}([0, \infty); L^2(\Omega))$$

satisfying the estimate

$$\sum_{k=0}^{m+1} \|D_t^k u(t)\|_{H_{m+1-k}} \le K$$

for some $K > 0$. We denote by D_m^0 the set of pairs (u_0, u_1), which satisfy the compatibility conditions of the mth order.

Throughout this paper, $E(t)$ is the energy of the solution at time t to problem (1.1)–(1.3) defined by

$$E(t) = \frac{1}{2}\left\{\|u_t\|^2 + \int_\Omega F(|\nabla u|^2)\mathrm{d}x\right\},$$

where the function F is defined by $F(s) = \int_0^s \sigma(\eta)\mathrm{d}\eta$.

By Hypothesis II, we have the following conditions on the function F:

(F1): $F(|\nabla u|^2) \ge k_0|\nabla u|^2$ if $|\nabla u|^2 < L$, where $k_0 = \sup_t \min_{0\le\theta\le|\nabla u|^2} \sigma(\theta)$.

(F2): There exists a constant $\tilde{k} > 0$ such that $\sigma(|\nabla u|^2)|\nabla u|^2 \ge \tilde{k}F(|\nabla u|^2)$ if $|\nabla u|^2 < L$, where $\tilde{k} = \sigma(|\nabla u|^2)/\sup\sigma(\theta)$, $0 < \theta < |\nabla u|^2$.

Suppose that

$$\sup_{0\le t\le T} \|\nabla u(t)\|_\infty^2 < L.$$

Since $\sigma(v)$ belongs to $C^{m+2}([0,L])$ for some $L>0$, there exist $M=\sup_t \max_{0\le\theta\le|\nabla u|^2}\sigma(\theta)$. By (F1) and (F2), we get the following inequality

$$\tilde{k}F(|\nabla u|^2) \le \sigma(|\nabla u|^2)|\nabla u|^2 \le M|\nabla u|^2 \le \frac{M}{k_0}F(|\nabla u|^2). \tag{2.2}$$

The main result of this paper reads as follows:

Theorem 2.2. *Let m be a positive integer and $(u_0,u_1)\in H^{m+1}(\Omega)\times H^m(\Omega)$ satisfy the compatibility of the mth order relative to problem (1.1)–(1.3). Suppose that Hypotheses I and II are satisfied with m and $p>0$ such that $m\ge N/2+2$ and $(2m-N-1)p>N$, there exist $x_0\in\mathbb{R}^N(N\ge 2)$ such that $\mathrm{dist}(x_0,\bar{\Omega})\ge 1$ and $\|\nabla\sigma(|\nabla u|^2)\|_\infty\le\delta$ with small $\delta>0$. Then, for the solution $u(t)\in X_m$ and a given $\varepsilon\in(0,1)$, we have the energy decay estimate*

$$E(t)=\mathcal{O}\left(t^{-\frac{2m(1+\varepsilon)+N(1-\varepsilon)}{(1-\varepsilon)N}}\right). \tag{2.3}$$

Remark 2.1. We can obtain the energy decay property depending on dimension N and parameter p by choosing a proper $\varepsilon\in(0,1)$. For example, if $\varepsilon=p/(p+N)$, then $E(t)=\mathcal{O}(t^{-(1+2m(2p+N)/N^2)})$ and if $\varepsilon=N/(p+N)$, then $E(t)=\mathcal{O}(t^{-(1+2m(p+2N)/pN)})$.

3. Proof of Theorem 2.2

First, we shall derive the inequality

$$TE(t+T)\le C'\left\{D(t)^2+\int_t^{t+T}\int_\Omega |u|^2\mathrm{d}x\mathrm{d}s+\int_t^{t+T}\int_\omega a(x)|\nabla u_t||\nabla u|\mathrm{d}x\mathrm{d}s\right\}. \tag{3.1}$$

The proof of (3.1) is based on the standard multipliers technique [13]. We sketch it briefly.

Multiplying Eq. (1.1) by u_t and integrating, then we get

$$E(t+T)+\int_t^{t+T}\int_\Omega a(x)|\nabla u_t|^2\mathrm{d}x\mathrm{d}s=E(t). \tag{3.2}$$

Multiplying Eq. (1.1) by $\eta(x)u$ and integrating, we obtain

$$\begin{aligned}\int_t^{t+T}\int_\Omega \sigma(|\nabla u|^2)|\nabla u|^2\eta\mathrm{d}x\mathrm{d}s &= -(u_t,\eta u)\big|_t^{t+T}+\int_t^{t+T}\int_\Omega \eta|u_t|^2\mathrm{d}x\mathrm{d}s-\int_t^{t+T}\int_\Omega\sigma(|\nabla u|^2)\nabla u\cdot\nabla\eta u\mathrm{d}x\mathrm{d}s\\ &\quad -\int_t^{t+T}\int_\Omega a(x)\nabla u_t\cdot\nabla\eta u\mathrm{d}x\mathrm{d}s-\int_t^{t+T}\int_\Omega a(x)\eta\nabla u_t\cdot\nabla u\mathrm{d}x\mathrm{d}s,\end{aligned} \tag{3.3}$$

where $\eta\in C^1(\bar{\Omega})$. Multiplying Eq. (1.1) by $h(x)\cdot\nabla u$ and integrating, we have

$$\begin{aligned}&\frac12\int_t^{t+T}\int_\Omega \nabla\cdot h|u_t|^2\mathrm{d}x\mathrm{d}s-\frac12\int_t^{t+T}\int_\Omega(\nabla\cdot h+h\cdot\nabla)\sigma(|\nabla u|^2)|\nabla u|^2\mathrm{d}x\mathrm{d}s+\int_t^{t+T}\int_\Omega\sum_{i,k=1}^N u_{x_i}u_{x_k}\frac{\partial}{\partial x_i}h_k\mathrm{d}x\mathrm{d}s\\ &\quad=-(u_t,h\cdot\nabla u)\big|_t^{t+T}+\int_t^{t+T}\int_\Omega\nabla\cdot(a(x)\nabla u_t)h\cdot\nabla u\mathrm{d}x\mathrm{d}s+\frac12\int_t^{t+T}\int_{\partial\Omega}\sigma(|\nabla u|^2)\left|\frac{\partial u}{\partial\nu}\right|^2\nu\cdot h\mathrm{d}\Gamma\mathrm{d}s.\end{aligned} \tag{3.4}$$

Setting $h(x)=\phi(r)(x-x_0)$, $r=|x-x_0|$ in (3.4) we obtain

$$\begin{aligned}&\frac12\int_t^{t+T}\int_\Omega(N\phi(r)+\phi'(r)r)|u_t|^2\mathrm{d}x\mathrm{d}s-\frac12\int_t^{t+T}\int_\Omega(N\phi(r)+\phi'(r)r)\sigma(|\nabla u|^2)|\nabla u|^2\mathrm{d}x\mathrm{d}s\\ &\quad-\frac12\int_t^{t+T}\int_\Omega\phi(r)(x-x_0)\cdot\nabla\sigma(|\nabla u|^2)|\nabla u|^2\mathrm{d}x\mathrm{d}s\\ &\quad+\int_t^{t+T}\int_\Omega\sigma(|\nabla u|^2)(\phi'(r)r^{-1}|(x-x_0)\cdot\nabla u|^2+\phi(r)|\nabla u|^2)\mathrm{d}x\mathrm{d}s\\ &\quad=-(u_t,\phi(x-x_0)\cdot\nabla u)\big|_t^{t+T}+\int_t^{t+T}\int_\Omega\nabla\cdot(a(x)\nabla u_t)\phi(r)(x-x_0)\cdot\nabla u\mathrm{d}x\mathrm{d}s\\ &\quad+\frac12\int_t^{t+T}\int_{\partial\Omega}\sigma(|\nabla u|^2)\left|\frac{\partial u}{\partial\nu}\right|^2\nu\cdot\phi(r)(x-x_0)\mathrm{d}\sigma\mathrm{d}s.\end{aligned} \tag{3.5}$$

From now on, we consider a function $\eta \in C^1(\bar{\Omega})$ such that $0 \le \eta \le 1$, $\eta = 1$ on $\hat{\omega}$ and $\eta = 0$ on Ω/ω, where $\hat{\omega}$ is an open set in $\bar{\Omega}$ with $\Gamma(x_0) \subset \Omega \cap \hat{\omega} \subset \omega$. Combining (3.3) and (3.5), we obtain for any $\alpha > 0$,

$$\begin{aligned}
&\int_t^{t+T}\int_\Omega \left(\left(\frac{N\phi(r)+\phi'(r)r}{2}\right) - \alpha\eta(x)\right)|u_t|^2 \mathrm{d}x\mathrm{d}s \\
&\quad + \int_t^{t+T}\int_\Omega \left(\alpha\eta(x) - \left(\frac{(N-2)\phi(r)+\phi'(r)r}{2}\right)\right)\sigma(|\nabla u|^2)|\nabla u|^2 \mathrm{d}x\mathrm{d}s \\
&\le \int_t^{t+T}\int_\omega \alpha a(x)|\nabla u_t||\nabla u|\mathrm{d}x\mathrm{d}s + \int_\Omega \alpha\eta(x)|u_t(t)||u(t)|\mathrm{d}x + \int_\Omega \alpha\eta(x)|u_t(t+T)||u(t+T)|\mathrm{d}x \\
&\quad + \int_t^{t+T}\int_\Omega \alpha\sigma(|\nabla u|^2)|\nabla u||\nabla\eta||u|\mathrm{d}x\mathrm{d}s + \int_t^{t+T}\int_\Omega \alpha a(x)|\nabla u_t||\nabla\eta||u|\mathrm{d}x\mathrm{d}s \\
&\quad + \frac{1}{2}\int_t^{t+T}\int_\Omega \phi(r)(x-x_0)\cdot\nabla\sigma(|\nabla u|^2)|\nabla u|^2\mathrm{d}x\mathrm{d}s \\
&\quad + \int_\Omega |u_t(t)||\phi(r)(x-x_0)\cdot\nabla u(t)|\mathrm{d}x + \int_\Omega |u_t(t+T)||\phi(r)(x-x_0)\cdot\nabla u(t+T)|\mathrm{d}x \\
&\quad + \left|-\int_t^{t+T}\int_\Omega \sigma(|\nabla u|^2)\phi'(r)r^{-1}|(x-x_0)\cdot\nabla u|^2\mathrm{d}x\mathrm{d}s\right| \\
&\quad + \left|\int_t^{t+T}\int_\Omega \nabla\cdot(a(x)\nabla u_t)\phi(r)(x-x_0)\cdot\nabla u\mathrm{d}x\mathrm{d}s\right| \\
&\quad + \frac{1}{2}\int_t^{t+T}\int_{\partial\Omega} \sigma(|\nabla u|^2)\left|\frac{\partial u}{\partial\nu}\right|^2 \nu\cdot\phi(r)(x-x_0)\mathrm{d}\sigma\mathrm{d}s \\
&= \int_t^{t+T}\int_\omega \alpha a(x)|\nabla u_t||\nabla u|\mathrm{d}x\mathrm{d}s + I_1+I_2+I_3+I_4+I_5+I_6+I_7+I_8+I_9+I_{10}.
\end{aligned} \tag{3.6}$$

First, we find a proper estimate for the term of the left hand side of (3.6). Now, we take a function $\phi(r)$, $r = |x-x_0|$, defined by

$$\phi(r) = \begin{cases} \delta_0 r^{1-N}, & N \ge 3, \\ \delta_0 r^{-\frac{3}{2}}, & N = 2, \end{cases} \tag{3.7}$$

where $\delta_0 > 0$, $r \ge 1$, and $\mathrm{dist}(x_0, \bar{\Omega}) \ge 1$. For case $N \ge 3$, we get

$$\begin{aligned}
&\int_t^{t+T}\int_\Omega \left(\left(\frac{N\phi(r)+\phi'(r)r}{2}\right) - \alpha\eta(x)\right)|u_t|^2 \mathrm{d}x\mathrm{d}s \\
&\quad + \int_t^{t+T}\int_\Omega \left(\alpha\eta(x) - \left(\frac{(N-2)\phi(r)+\phi'(r)r}{2}\right)\right)\sigma(|\nabla u|^2)|\nabla u|^2 \mathrm{d}x\mathrm{d}s \\
&= \int_t^{t+T}\int_\Omega \left(\frac{1}{2}\delta_0 r^{1-N} - \alpha\eta\right)|u_t|^2\mathrm{d}x\mathrm{d}s + \int_t^{t+T}\int_\Omega \left(\alpha\eta - \frac{1}{2}\delta_0 r^{1-N}(-1)\right)\sigma(|\nabla u|^2)|\nabla u|^2\mathrm{d}x\mathrm{d}s.
\end{aligned}$$

Taking $0 < \alpha < \delta_0 r^{1-N}/2$, then we get the following inequality.

$$\begin{aligned}
&\int_t^{t+T} 2C_\alpha \tilde{k}E(s)\mathrm{d}s \le \int_t^{t+T}\int_\Omega 2C_\alpha \frac{1}{2}\left(|u_t|^2 + \tilde{k}F(|\nabla u|^2)\right)\mathrm{d}x\mathrm{d}s \\
&\le \int_t^{t+T}\int_\Omega 2C_\alpha \frac{1}{2}\left(|u_t|^2 + \sigma(|\nabla u|^2)|\nabla u|^2\right)\mathrm{d}x\mathrm{d}s \\
&\le \int_t^{t+T}\int_\Omega \left(\left(\frac{N\phi(r)+\phi'(r)r}{2}\right) - \alpha\eta(x)\right)|u_t|^2\mathrm{d}x\mathrm{d}s \\
&\quad + \int_t^{t+T}\int_\Omega \left(\alpha\eta(x) - \left(\frac{(N-2)\phi(r)+\phi'(r)r}{2}\right)\right)\sigma(|\nabla u|^2)|\nabla u|^2\mathrm{d}x\mathrm{d}s,
\end{aligned} \tag{3.8}$$

where $C_\alpha = (\delta_0 r^{1-N}/2 - \alpha\eta) > 0$. In the first inequality, we have used the fact $\tilde{k} \le 1$. In the second inequality we used condition (F2). In the third inequality we have used the fact $C_\alpha = \delta_0 r^{1-N}/2 - \alpha\eta = (N\phi(r) + \phi'(r)r)/2 - \alpha\eta(x) < \alpha\eta(x) - ((N-2)\phi(r) + \phi'(r)r)/2$.

Second, we will find a proper estimate for the term of the right hand side of (3.6).

$$\begin{aligned} I_1 &= \int_\Omega \alpha\eta|u_t(t)||u(t)|\mathrm{d}x \\ &\le \frac{1}{4}\delta_0\int_\Omega |u_t(t)|^2\mathrm{d}x + \frac{1}{4}\delta_0 C_1\frac{1}{k_0}\int_\Omega F(|\nabla u(t)|^2)\mathrm{d}x \le C_2E(t), \end{aligned}$$

where $C_2 = \max\{\delta_0/2, \delta_0 C_1/2k_0\}$ and we have used Poincare's inequality $\|u\|_{L^2(\Omega)} \le C_1\|\nabla u\|_{L^2(\Omega)}$ for some constant $C_1 > 0$, the facts $0 \le \eta \le 1$ and $\alpha < \delta_0 r^{1-N}/2 < \delta_0/2 (N \ge 3, r \ge 1)$, and the condition (F1). Similarly, we obtain

$$I_2 = \int_\Omega \alpha\eta|u_t(t+T)||u(t+T)|\mathrm{d}x \le C_2E(t+T).$$

Next, we estimate I_3 given by

$$\begin{aligned} I_3 &= \int_t^{t+T}\int_\Omega \alpha\sigma(|\nabla u|^2)|\nabla u||\nabla\eta \parallel u|\mathrm{d}x\mathrm{d}s \le \int_t^{t+T}\int_\Omega M\frac{1}{2}\delta_0C^*|\nabla u||u|\mathrm{d}x\mathrm{d}s \\ &\le \frac{1}{2}\delta_0^2C^{*2}M^2\frac{1}{\varepsilon}\int_t^{t+T}\int_\Omega |u|^2\mathrm{d}x\mathrm{d}s + \varepsilon\frac{1}{2}\int_t^{t+T}\int_\Omega |\nabla u|^2\mathrm{d}x\mathrm{d}s \\ &\le \frac{1}{2}\delta_0^2M^2C^{*2}\frac{1}{\varepsilon}\int_t^{t+T}\int_\Omega |u|^2\mathrm{d}x\mathrm{d}s + \frac{1}{k_0}\varepsilon\int_t^{t+T}\frac{1}{2}\left(\|u_t(s)\|^2 + \int_\Omega F(|\nabla u|^2)\mathrm{d}x\right)\mathrm{d}s \\ &\le C_3(\varepsilon)\int_t^{t+T}\int_\Omega |u|^2\mathrm{d}x\mathrm{d}s + \frac{\varepsilon}{k_0}\int_t^{t+T}E(s)\mathrm{d}s, \end{aligned}$$

where $C_3(\varepsilon) = \delta_0^2C^{*2}M^2/2\varepsilon$. Here, we have used the facts $0 < \alpha < \delta_0/2$, $0 \le \eta \le 1$, $|\nabla\eta| \le C^*$, $M = \sup_t \max_{0\le\theta\le|\nabla u|^2} \sigma(\theta)$ and the condition (F1). We also get

$$\begin{aligned} I_4 &= \int_t^{t+T}\int_\Omega \alpha a(x)|\nabla u_t||\nabla\eta||u|\mathrm{d}x\mathrm{d}s \\ &\le \int_t^{t+T}\int_\Omega \frac{1}{2}\delta_0C^*a(x)|\nabla u_t||u|\mathrm{d}x\mathrm{d}s \\ &\le \frac{1}{2}\delta_0^2C^{*2}\frac{1}{\varepsilon}\int_t^{t+T}\int_\Omega a(x)|\nabla u_t|^2\mathrm{d}x\mathrm{d}s + \frac{1}{2}\varepsilon\int_t^{t+T}\int_\Omega a(x)|u|^2\mathrm{d}x\mathrm{d}s \\ &\le \frac{1}{2}\delta_0^2C^{*2}\frac{1}{\varepsilon}\int_t^{t+T}\int_\Omega a(x)|\nabla u_t|^2\mathrm{d}x\mathrm{d}s + \frac{1}{2}\|a\|_\infty\varepsilon\int_t^{t+T}\int_\Omega |u|^2\mathrm{d}s\mathrm{d}x \\ &\le \frac{1}{2}\delta_0^2C^{*2}\frac{1}{\varepsilon}\int_t^{t+T}\int_\Omega a(x)|\nabla u_t|^2\mathrm{d}x\mathrm{d}s + \|a\|_\infty\frac{C_1}{k_0}\varepsilon\int_t^{t+T}\frac{1}{2}\left(\|u_t(s)\|^2 + \int_\Omega F(|\nabla u|^2)\mathrm{d}x\right)\mathrm{d}s \\ &\le C_4(\varepsilon)\int_t^{t+T}\int_\Omega a(x)|\nabla u_t|^2\mathrm{d}x\mathrm{d}s + \varepsilon C_5\int_t^{t+T}E(s)\mathrm{d}s, \end{aligned}$$

where $C_4(\varepsilon) = \delta_0^2C^{*2}/2\varepsilon$, $C_5 = \|a\|_\infty C_1/k_0$. Using the facts $\sup_t \|\nabla\sigma(|\nabla u|^2)\|_\infty \le \delta$, $|\phi| \le \delta_0$ and $1 \le |x-x_0| \le C(x_0) = \sup_{x\in\Omega}|x-x_0|$, we get

$$I_5 = \frac{1}{2}\int_t^{t+T}\int_\Omega \phi(r)|x-x_0||\nabla\sigma(|\nabla u|^2)||\nabla u|^2\,\mathrm{d}x\mathrm{d}s \le C_7\delta\int_t^{t+T}E(s)\mathrm{d}s,$$

where $C_7 = \delta_0C(x_0)/k_0$. It is easy to check the following two inequalities,

$$I_6 = \int_\Omega |u_t(t)||\phi(r)(x-x_0)\cdot\nabla u(t)|\mathrm{d}x \le C_7E(t),$$

$$I_7 = \int_\Omega |u_t(t+T)||\phi(r)(x-x_0)\cdot\nabla u(t+T)|\mathrm{d}x \le C_7E(t+T).$$

For estimating I_8, we will use the facts $\sup_t \|\sigma(|\nabla u|^2)\|_\infty \le \gamma$ and $\left|\phi'(r)r^{-1}|x-x_0|^2\right| \le C^{**}$. In fact, we get

$$\begin{aligned} I_8 &= \left|-\int_t^{t+T}\int_\Omega \sigma(|\nabla u|^2)\phi'(r)r^{-1}|(x-x_0)\cdot\nabla u|^2\mathrm{d}x\mathrm{d}s\right| \\ &\le \gamma C^{**}\int_t^{t+T}\int_\Omega |\nabla u|^2\mathrm{d}x\mathrm{d}s \\ &\le \gamma C^{**}\frac{1}{k_0}\int_t^{t+T}E(s)\mathrm{d}s. \end{aligned}$$

Using the facts $a(x) = 0$ on Ω/ω, the regularity of u, $|\nabla\phi(r)(x-x_0)| \le C_8$ and $|\frac{1}{|\nabla u|}\phi(r)(x-x_0)\Delta u| \le C_9$ we get,

$$\begin{aligned} I_9 &= \left|\int_t^{t+T}\int_\Omega \nabla\cdot(a(x)\nabla u_t)\phi(r)(x-x_0)\cdot\nabla u\mathrm{d}x\mathrm{d}s\right| \\ &\le \left|\int_t^{t+T}\int_\omega a(x)\nabla u_t(\nabla\cdot(\phi(r)(x-x_0)))\cdot\nabla u\mathrm{d}x\mathrm{d}s\right| + \left|\int_t^{t+T}\int_\omega a(x)\nabla u_t\phi(r)(x-x_0)\Delta u\mathrm{d}x\mathrm{d}s\right| \\ &\le C_8\int_t^{t+T}\int_\omega a(x)|\nabla u_t\|\nabla u|\mathrm{d}x\mathrm{d}s + C_9\int_t^{t+T}\int_\omega a(x)|\nabla u_t\|\nabla u|\mathrm{d}x\mathrm{d}s. \end{aligned}$$

Finally, we will estimate I_{10}. For this estimation, we again use (3.5). This time, we take a vector field $h = \phi(r)(x-x_0)$ such that

$$h\cdot\nu \ge 0, \qquad h = \nu \quad \text{on } \Gamma(x_0) \text{ and supp } h \subset \hat{\omega}.$$

Then, we can derive

$$\begin{aligned} I_{10} &= \frac{1}{2}\int_t^{t+T}\int_{\partial\Omega}\sigma(|\nabla u|^2)\left|\frac{\partial u}{\partial\nu}\right|^2 \nu\cdot\phi(r)(x-x_0)\mathrm{d}\sigma\,\mathrm{d}s \\ &\le \frac{1}{2}\delta\delta_0 C(x_0)\int_t^{t+T}\int_{\Gamma(x_0)}\left|\frac{\partial u}{\partial\nu}\right|^2\mathrm{d}\Gamma\mathrm{d}s \\ &\le C_{10}\Bigg(E(t)+E(t+T)+\int_t^{t+T}\int_{\hat{\omega}}|u_t|^2\mathrm{d}x\mathrm{d}s + \int_t^{t+T}\int_{\hat{\omega}}\sigma(|\nabla u|^2)|\nabla u|^2\mathrm{d}x\mathrm{d}s \\ &\quad + \int_t^{t+T}\int_{\hat{\omega}}|\nabla\sigma(|\nabla u|^2)||\nabla u|^2\mathrm{d}x\mathrm{d}s + \int_t^{t+T}\int_{\hat{\omega}}\sigma(|\nabla u|^2)|\nabla u|^2\mathrm{d}x\mathrm{d}s \\ &\quad + \int_t^{t+T}\int_{\hat{\omega}}|\nabla u|^2\mathrm{d}x\mathrm{d}s + \int_t^{t+T}\int_{\hat{\omega}}a(x)|\nabla u_t\|\nabla u|\mathrm{d}x\mathrm{d}d\Bigg) \\ &\le C\left(E(t)+E(t+T)+\int_t^{t+T}\int_\omega a(x)|\nabla u_t|^2\mathrm{d}x\mathrm{d}s + \int_t^{t+T}\int_\omega a(x)|\nabla u_t\|\nabla u|\mathrm{d}x\mathrm{d}s\right) \end{aligned}$$

where we used I_9, the fact $\int_t^{t+T}\int_\omega|\nabla u_t|^2\mathrm{d}x\mathrm{d}s$ will be absorbed into $\int_t^{t+T}\int_\Omega a(x)|\nabla u_t|^2\mathrm{d}x\mathrm{d}s$ and the fact (from (3.3))

$$\int_t^{t+T}\int_{\hat{\omega}}|\nabla u|^2\mathrm{d}x\mathrm{d}s \le CE(t)+CE(t+T)+C'\int_t^{t+T}\int_\omega a(x)|\nabla u_t|^2\mathrm{d}x\mathrm{d}s + C'\int_t^{t+T}\int_\omega a(x)|\nabla u_t||\nabla u|\mathrm{d}x\mathrm{d}s.$$

Here we also consider a function $\eta \in C^1(\bar{\Omega})$ such that $0 \le \eta \le 1$, $\eta = 1$ on $\hat{\omega}$ and $\eta = 0$ on Ω/ω, where $\hat{\omega}$ is an open set in $\bar{\Omega}$ with $\Gamma(x_0) \subset \Omega\cap\hat{\omega} \subset \omega$.

Setting

$$D(t)^2 \equiv E(t)-E(t+T) = \int_t^{t+T}\int_\Omega a(x)|\nabla u_t|^2\mathrm{d}x\mathrm{d}s, \tag{3.9}$$

and combining (3.6), (3.8), (3.9) and inequalities $I_1 \sim I_{10}$, we can derive the useful inequality

$$\begin{aligned} &\int_t^{t+T}\left(2\tilde{k}C_\alpha - \left(\frac{1}{k_0}+C_5\right)\varepsilon - C_7\delta - \frac{C^{**}}{k_0}\gamma\right)E(s)\mathrm{d}s \\ &\le C''\left\{D^2(t)+E(t+T)+\int_t^{t+T}\int_\Omega|u|^2\mathrm{d}x\mathrm{d}s + \int_t^{t+T}\int_\omega a(x)|\nabla u_t||\nabla u|\right\}\mathrm{d}x\mathrm{d}s. \end{aligned} \tag{3.10}$$

It is noted that we can choose sufficient small $\varepsilon > 0$, $\delta > 0$, $\gamma > 0$ so that $2\tilde{k}C_\alpha - \left(\frac{1}{k_0}+C_5\right)\varepsilon - C_7\delta - \frac{C^{**}}{k_0}\gamma > 0$ for $N \ge 3$. Therefore, we can choose proper constant $C(\delta_0) > 0$ such that

$$\int_t^{t+T} C(\delta_0)E(s)\mathrm{d}s \le \int_t^{t+T}\left(2\tilde{k}C_\alpha - \left(\frac{1}{k_0}+C_5\right)\varepsilon - C_7\delta - \frac{C^{**}}{k_0}\gamma\right)E(s)\mathrm{d}s.$$

Using the above inequality and (3.10), we get the following inequality

$$\int_t^{t+T} C(\delta_0)E(s)\mathrm{d}s \le C''\left\{D^2(t)+E(t+T)+\int_t^{t+T}\int_\Omega|u|^2\mathrm{d}x\mathrm{d}s + \int_t^{t+T}\int_\omega a(x)|\nabla u_t||\nabla u|\right\}\mathrm{d}x\mathrm{d}s.$$

Combining this result and $TE(t+T) \le \int_t^{t+T} E(s)\mathrm{d}s$, we can obtain inequality (3.1). For case $N = 2$, we take $0 < \alpha < \delta_0 r^{-3/2}/8$ and $C_\alpha = \delta_0 r^{-3/2}/8 - \alpha\eta$ and using the similar method above we get the same result. Finally, we must show the boundedness of the second term in the right hand side of (3.1). For this, we have the following proposition, which is proved by a similar way due to [8,17].

Proposition 3.1. *Let $T > 0$ be a fixed and sufficient large number, and assume that $C_0 \equiv C_0(K) > 0$ and $\varepsilon > 0$ such that $C(K)E(0) < \varepsilon$. Then there exists a constant $C''' > 0$ independent of (u_0, u_1) such that the estimate*

$$\int_t^{t+T} \|u(s)\|^2 \mathrm{d}s \le C''' \left\{ \int_t^{t+T} \int_\Omega a(x)|\nabla u_t|^2 \mathrm{d}x\mathrm{d}s + \int_t^{t+T} \int_\omega |u_t|^2 \mathrm{d}x\mathrm{d}s \right\}$$

holds for any energy finite solution to problem (1.1)–(1.3).

Proof. We sketch it briefly. Suppose that the assertion was false. Then there exists a sequence $\{t_n\} \subset R$ and a sequence of solutions $\{u_n\}$ such that

$$\int_{t_n}^{t_n+T} \|u_n(s)\|^2 \mathrm{d}s = 1$$

and

$$\int_{t_n}^{t_n+T} \int_\Omega a(x)|\nabla u_{nt}|^2 \mathrm{d}x\mathrm{d}s + \int_{t_n}^{t_n+T} \int_\omega |u_{nt}|^2 \mathrm{d}x\mathrm{d}s \to 0$$

as $n \to \infty$. Setting $v_n(t) = u_n(t + t_n)$, we have

$$\int_0^T \|v_n(s)\|^2 \mathrm{d}s = 1$$

and

$$\int_0^T \int_\Omega a(x)|\nabla v_{nt}|^2 \mathrm{d}x\mathrm{d}s + \int_0^T \int_\omega |v_{nt}|^2 \mathrm{d}x\mathrm{d}s \to 0$$

as $n \to \infty$, and by (3.2) we also get

$$\begin{aligned}\sup_{0\le t\le T} \left\{ \|v_{nt}(t)\|^2 + \|\nabla v_n(t)\|^2 \right\} &\le \sup_{0\le t\le T} \left\{ \|v_{nt}(t)\|^2 + \int_\Omega F(|\nabla v_n(t)|^2)\mathrm{d}x \right\} \\ &= 2E(v_n(0)) = 2\left\{ E(v_n(T)) + \int_0^T \int_\Omega a(x)|\nabla v_{nt}|^2 \mathrm{d}x\mathrm{d}s \right\} \le C_1 < \infty\end{aligned}$$

for large n, where C_1 is a constant independent of (u_0, u_1). Hence, there $v \in L^\infty([0,T]; H_0^1(\Omega)) \cap W^{1,\infty}([0,T]; L^2(\Omega))$ such that $v_n \to v$ weakly* in $L^\infty([0,T]; H_0^1(\Omega)) \cap W^{1,\infty}([0,T]; L^2(\Omega))$ and strongly in $L^2(\Omega \times [0,T])$. Thus from the equation

$$v_{ntt} - \nabla \cdot \{\sigma(|\nabla v_n|^2)\nabla v_n\} - \nabla \cdot (a(x)\nabla v_{nt}) = 0 \quad \text{in } \Omega \times [0,T]$$

we have

$$v_{tt} - \nabla \cdot \{\sigma(|\nabla v|^2)\nabla v\} = 0 \quad \text{in } \Omega \times [0,T] \tag{3.11}$$

with the additional conditions

$$\int_0^T \int_\Omega a(x)|\nabla v_t|^2 \mathrm{d}x\mathrm{d}s = 0, \qquad \int_0^T \int_\omega |v_t|^2 \mathrm{d}x\mathrm{d}s = 0 \tag{3.12}$$

and

$$\int_0^T \|v(s)\|^2 \mathrm{d}s = 1. \tag{3.13}$$

From the second part of (3.12), we see $v_t = 0$ in $\omega \times [0,T]$.

Using a similar method in [8, Lemma 4.1], we can check

$$\left| \frac{\partial}{\partial t}\sigma(|\nabla v|^2) \right| + |\nabla \sigma(|\nabla v|^2)| + \left| \nabla \frac{\partial}{\partial t}\sigma(|\nabla v|^2) \right| \le \varepsilon.$$

By the unique continuation property for the wave equation, there exists $T_0 > 0$ such that if $T > T_0$, then $v_t(x,t) = 0$ on $\Omega \times [0,T]$ which together with (3.11) implies that $v(x,t) = v(x)$, independent of t and $-\Delta v(x) = 0$ in Ω. Since $v \in C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))$, we can see $\nabla v(x) = 0$ and so $v(x) \equiv 0$. This is a contradiction to (3.13). $\square$

Using (3.1) and Proposition 3.2, we can derive the useful result for the energy decay estimates.

Proposition 3.2. *Let $T > 0$ be a fixed and sufficient large number. For the solution $u(t) \in X^m$, there exists a constant $C > 0$ satisfying the estimate*

$$E(t+T) \leq C \left\{ D(t)^2 + \int_t^{t+T} \int_\omega a(x)|\nabla u_t|\,|\nabla u| \mathrm{d}x\mathrm{d}s \right\}. \tag{3.14}$$

Proof. Form (3.1), using Proposition 3.1 and Poincare's inequality, we get

$$TE(t+T) \leq C'''' \left\{ D(t)^2 + \int_t^{t+T} \int_\omega |\nabla u_t|^2 \mathrm{d}x\mathrm{d}s + \int_t^{t+T} \int_\omega a(x)|\nabla u_t||\nabla u| \mathrm{d}x\mathrm{d}s \right\}.$$

Since $\int_t^{t+T} \int_\omega |\nabla u_t|^2 \mathrm{d}x\mathrm{d}s$ will be absorbed into $\int_t^{t+T} \int_\Omega a(x)|\nabla u_t|^2 \mathrm{d}x\mathrm{d}s \equiv D^2(t)$, we obtain (3.14). □

We are now in a position to prove the main result. In order to complete the proof of Theorem 2.2, we must find a proper estimate for the term of the first integral in the right hand side of (3.14). By the Gagliardo–Nirenberg inequality, we have

$$\|\nabla u\|_{L^2(\omega)} \leq C \|\nabla u\|_{H^m(\omega)}^{\theta} \|\nabla u\|_{L^{1+\varepsilon}(\omega)}^{1-\theta}, \tag{3.15}$$

where $\theta = (1-\varepsilon)N/(2m(1+\varepsilon)+N(1-\varepsilon))$, $0 < \varepsilon < 1$. Using hypothesis on F and Hölder inequality we get

$$\begin{aligned} \int_\omega |\nabla u|^{1+\varepsilon} \mathrm{d}x &\leq C_1 \left(\int_\omega \mathrm{d}x \right)^{\frac{1-\varepsilon}{2}} \left(\int_\omega |\nabla u|^{1+\varepsilon \frac{2}{1+\varepsilon}} \mathrm{d}x \right)^{\frac{1+\varepsilon}{2}} \\ &\leq C_1 \left(\int_\omega \mathrm{d}x \right)^{\frac{1-\varepsilon}{2}} \left(\int_\omega \frac{1}{k_0} F(|\nabla u|^2) \mathrm{d}x \right)^{\frac{1+\varepsilon}{2}} \\ &\leq CE(t)^{\frac{1+\varepsilon}{2}}. \end{aligned} \tag{3.16}$$

From (3.15) and (3.16), we obtain

$$\begin{aligned} \int_t^{t+T} \int_\omega a(x)|\nabla u_t|\,|\nabla u| \mathrm{d}x\mathrm{d}s &\leq C \left(\int_t^{t+T} \int_\omega a(x)|\nabla u_t|^2 \mathrm{d}x\mathrm{d}s \right)^{\frac{1}{2}} \left(\int_t^{t+T} \int_\omega a(x)|\nabla u|^2 \mathrm{d}x\mathrm{d}s \right)^{\frac{1}{2}} \\ &\leq C\|a(\cdot)\|_\infty^{\frac{1}{2}} D(t) \|\nabla u\|_{L^2(\omega)} \leq C' D(t) \sup_{t \leq s \leq t+T} \|\nabla u(s)\|_{L^{1+\varepsilon}(\omega)}^{1-\theta} \\ &\leq C'' D(t) E(t)^{\frac{1-\theta}{2}} \equiv A(t)^2. \end{aligned}$$

Thus we have from (3.14) $E(t+T) \leq C(D(t)^2 + A(t)^2)$. Since $E(t) - E(t+T) \equiv D(t)^2$, we get $E(t) \leq C(D(t)^2 + A(t)^2)$. Using the definition of $A(t)^2$ and Young's inequality we get the inequality $E(t) \leq CD(t)^{\frac{2}{1+\theta}}$ or $E(t)^{1+\theta} \leq CD(t)^2$,, where $\theta = (1-\varepsilon)N/(2m(1+\varepsilon)+N(1-\varepsilon))$, $0 < \varepsilon < 1$. Applying Lemma 2.1 to the above inequality, we get the energy estimate (2.3), which completes the proof of Theorem 2.2.

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